# BINARY LANGUAGES AND BINARY THEORIES

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Abstract. We illustrate by an example the fact that a theory in a binary language may not be a binary theory.

## 1. Binary relations over the integers

This note is in the general context of model-theory. The reader interested in basic definitions may for instance read first chapter of [3].

**Definition 1.1.** *A language is said to be binary if all it relational symbols are binary and all its functional symbols are unary.*

*A theory $T$ in a language $\mathcal{L}$ is said to be binary if each $\mathcal{L}$-formula is equivalent in $T$ to a boolean combination of binary $\mathcal{L}$-formulas.*

It is proven in [4] that an o-minimal $\mathcal{L} \cup \{<\}$-theory expanding the theory of a dense lineary ordered set is binary if and only if there is a binary language $\mathcal{L}' \cup \{<\}$ in which the theory $T$ can be restated (each symbol in $\mathcal{L}'$ interpreting unary $\emptyset$-definable function).

In this note, we show an example of a theory in a binary language which is not binary.

Denote by $\mathcal{L}_2$ the (purely relational and binary) language where we put a binary relational symbol $\widetilde{S}$ for each subset $S$ of $\mathbb{N}^2$ and let $\mathcal{D} = \langle \mathbb{N}; \mathcal{L}_2 \rangle$ be the $\mathcal{L}_2$-structure where each symbol $\widetilde{S}$ is interpreted by the subset $S$ of $\mathbb{N}^2$.

We will show that the theory of $\mathcal{D}$ is not binary though its language is.

As we are interested in ternary relation definable in binary relational language, let first recall this theorem due to Julia Robinson :

**Theorem 1.2** ([5]). *The addition on the integers is first-order definable in the structure $\langle \mathbb{N}; S, | \rangle$ where $|$ denotes the binary relation of integer divisability and $S$ the unary "successor" function.*

Thus the addition is definable in $\mathcal{D}$; we will now prove it can not be defined by a quantifier free formula in this structure.

**Lemma 1.3.** *If the addition is definable by a quantifier-free formula in $\mathcal{D}$ then there exists a finite familly of sets $(Z_i \subseteq \mathbb{N}^2)_{i \in \{1,\ldots,n\}}$ such that*

- *$\{Z_i\}_{1 \leq i \leq n}$ covers $\mathbb{N}^2$ ,*
- *for each $i \in \{1, \ldots, n\}$, $(x, y) \in \mathbb{N}^2$, $k > 0$ with $(x, y) \in Z_i$, one either has $(x + k, y) \notin Z_i$ or $(x, y + k) \notin Z_i$.*

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*Proof.* Suppose that the addition is definable by a quantifier-free formula in $\mathcal{D}$. Consider a boolean combination $S(x,y,z)$ of terms in $\mathcal{L}_2$ such that $S(x,y,z) \leftrightarrow z = x + y$.

Chose this combination in a disjunctive normal form and note that each finite conjonction of symbols and negations of a symbol in which only two - say $x$ and $y$ - of the three variables appear can be replaced by a single symbol $\widetilde{S}(x,y)$.

Thus we can describe the graph of the addition as a formula

$$\bigvee_{i=1}^{n} \left(\widetilde{Z_i}(x,y) \wedge \widetilde{Y_i}(x,z) \wedge \widetilde{X_i}(y,z)\right)$$

where $X_i$'s, $Y_i$'s and $Z_i$'s denote subsets of $\mathbb{N}^2$.

Replace each $Z_i$ by $\{(x,y) \in Z_i \mid \exists z \left(\widetilde{Y_i}(x,z) \wedge \widetilde{X_i}(y,z)\right)\}$; it is clear that the new formula is equivalent to the old one.

By definition of the $X_i$'s, $Y_i$'s and $Z_i$'s, we know that $\bigcup_{i=1}^{n} Z_i = \mathbb{N}^2$ and for each $i$ we now have the implication

$$\widetilde{Z_i}(x,y) \to \exists z \left(\widetilde{Y_i}(x,z) \wedge \widetilde{X_i}(y,z)\right) \tag{1}$$

and the equivalence

$$\left(\widetilde{Z_i}(x,y) \wedge \widetilde{X_i}(x,z) \wedge \widetilde{Y_i}(y,z)\right) \leftrightarrow \left(\widetilde{Z_i}(x,y) \wedge x + y = z\right). \tag{2}$$

Fix an $i$ and suppose we have some $(x,y) \in Z_i$ and some $k \in \mathbb{N}$ such that $(x, y+k) \in Z_i$ and $(x+k, y) \in Z_i$

Then by (1) and the left-to-right implication in (2), we get from $(x, y+k) \in Z_i$ that $(x, x+y+k) \in X_i$.

Symetricaly we deduce $(y, x+y+k) \in Y_i$.

We now have $\widetilde{X_i}(x, x+y+k) \wedge \widetilde{Y_i}(y, x+y+k) \wedge \widetilde{Z_i}(x,y)$ which by the right-to-left implication in (2) implies $x + y = x + y + k$; hence $k = 0$ □

**Proposition 1.4.** *There does not exist any finite covering* $\{Z_i\}_{1 \leq i \leq n}$ *of* $\mathbb{N}^2$ *satisfying for each* $i \in \{1, \ldots, n\}$, $(x,y) \in \mathbb{N}^2$ *and* $k \neq 0$, *the implication*

$$(x,y) \in Z_i \to \left((x+k, y) \notin Z_i \text{ or } (x, y+k) \notin Z_i\right). \tag{3}$$

*Proof.* First, recall VAN DER WAERDEN Theorem

**Theorem 1.5** (see for instance [2])**.** *For any* $l \in \mathbb{N}^*$ *and for any* $k \in \mathbb{N}$, *there is an integer* $m$ *such that for any partition* $(P_1, \ldots, P_l)$ *of* $\mathbb{N}$ *and for any collection of* $m$ *distinct integers* $\{x_1, \ldots, x_m\}$ *in arithmetic progression, one can extract a collection of* $k$ *distinct integers* $\{x_{i_1}, \ldots, x_{i_k}\}$ *in arithmetic progression, such that all the extracted* $x_{i_j}$*'s belong to a single* $P_s$.

Denote $W(l,k)$ the VAN DER WAERDEN number (that is the smallest among all such $m$) ; we define by induction the numbers $N_1 = 1$ and $N_{l+1} = W(l+1, 1+N_l)$.

For $(\alpha, \beta) \in \mathbb{N}^2$, $L \in \mathbb{N}$ and $\gamma \in \mathbb{N}^*$, let $T(\alpha, \beta, \gamma, L)$ be the subset of $\mathbb{N}^2$ given by

$$T(\alpha, \beta, \gamma, L) := \{(\alpha + k\gamma, \beta + \gamma); (k,l) \in \mathbb{N}^2, k + l \leq L\}$$

(note that these triangular sets are stable by $+$).

We will prove by induction on $n$ that, for any $(\alpha, \beta) \in \mathbb{N}^2$ and $\gamma \in \mathbb{N}^*$ there is no covering of $T(\alpha, \beta, \gamma, N_n)$ into $n$ subsets $S_1$ , ..., $S_n$ satisfying

$$\big((x, y) \in S_i \text{ and } k > 0\big) \to \big((x + k, y) \notin S_i \text{ or } (x, y + k) \notin S_i\big).$$

- The result is clear for $n = 0$.
- Given a triangle $T(\alpha, \beta, \gamma, N_{n+1})$, suppose that there exists such a covering $(S_1, \ldots, S_{n+1})$.

  By van der Waerden Theorem, considering the partition

  $$\big((S_i \setminus \bigcup_{j<i} S_j) \cap \{(\alpha, \beta + k\gamma) \in \mathbb{N}^2\}_{k\in\{0,\ldots,N_n\}}\big)_{i\in\{1,\ldots,n\}}$$

  of $\{(\alpha, \beta + k\gamma) \in \mathbb{N}^2\}_{k\in\{0,\ldots,N_n\}}$, there exists a $i_0$, a $\gamma'$ and a $\alpha'$ such that for all $k \in \{0, \ldots, N_n\}$

  $$\alpha' + k\gamma' \leq N_{n+1} \text{ and } (\alpha + \gamma(\alpha' + k\gamma'), \beta) \in S_{i_0}.$$

  For simplicity, we will assume $i_0 = n + 1$.
  Let $\alpha'' = \alpha + \gamma\alpha$ , $\beta'' = \beta + \gamma\gamma'$ and $\gamma'' = \gamma\gamma'$.
  Due to the implication

  $$\big((x, y) \in S_i \text{ and } k > 0\big) \to \big((x + k, y) \notin S_i \text{ or } (x, y + k) \notin S_i\big),$$

  the points $(\alpha'' + k\gamma'', \beta'' + l\gamma'')$ do not belong to $S_{n+1}$, for all $k \in \mathbb{N}$ and $l \in \mathbb{N}^*$ for which $k + l \leq N_n$.

  The sets $S'_i = S_i \cap T(\alpha'', \beta'', \gamma''', N_n)$ , $1 \leq i \leq n$, would then
  - cover $T(\alpha'', \beta'', \gamma'', N_n)$ and
  - verify the implication $\big((x, y) \in S'_i \text{ and } k > 0\big) \to \big((x + k, y) \notin S'_i \text{ or } (x, y + k) \notin S'_i\big)$.

  This contradicts the step $n$ in the induction.

It is now clear that it is not possible to cover $\mathbb{N}^2$ by a finite family of sets $(S_i)_{i\in\{1\ldots n\}}$ satisfying (3). □

We thus have proved :

**Corollary 1.6.** *The theory of the structure $\mathcal{D}$ is not binary and does not admit quantifier elimination.*

**Remark 1.7.** *Note that the structure $\mathcal{D}$ defines all possible subsets of $\mathbb{N}^n$ and this for each $n$, for this structure defines every subset of $\mathbb{N}$ and also bijections between $\mathbb{N}$ and $\mathbb{N}^n$ (it is well known that there are such bijections definable in $\langle \mathbb{N}; +, \cdot \rangle$, thus in $\mathcal{D}$).*

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